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Math 131 - Spring 2023 - Final Exam

Print name:____

Section number:________ **Instructor's name:**________________________

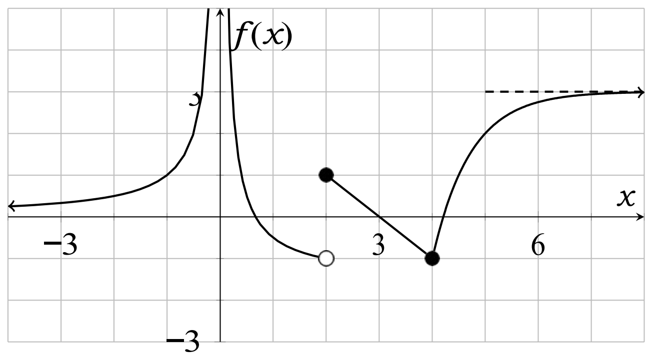
Directions:

- This exam has 12 questions.
- You will have two hours to complete this exam.
- It will be graded out of 106 points.
- Show your work. Answers (even correct ones) without the corresponding work will receive no credit.
- A formula sheet has been provided on the last page of this exam. You may not refer to any other notes or textbooks during the exam.
- You may use a calculator as long as it is unable to connect to the internet.
- You may only speak with your instructor during the exam.

Good luck!

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	12	6	6	6	14	10	4	10	10	10	10	8	106
Score:													

1. [12 points] Using the graph below, find each of the following. If the answer does not exist, write "DNE". The dotted line on the right of the graph represents a horizontal asymptopte and is not a part of the function.



(a) [2 points] $f(2)$
$f(2) = 1$

(b) [2 points] $\lim_{x \to 0} f(x)$

$\lim_{x \to 0} f(x) = \infty$

(c) [2 points] $\lim_{x \to \infty} f(x)$
$\lim_{x \to \infty} f(x) = 3$

(d) [2 points] $\lim_{x \to 2^-} f(x) = -1$

(e) [2 points] $\lim_{x \to 2} f(x)$ DNE

(f) [2 points] $\lim_{x \to 4} f(x)$ $= -1$

2. [6 points] Suppose $g(x) = x^2 - 1$.

(a) [1 point] Find the value of $g(4)$.

$$g(4) = 4^2 - 1 = 15$$

(b) [2 points] Simplify completely: $g(4 + h)$.

$$g(4+h) = (4+h)^2 - 1 = (16 + 8h + h^2) - 1$$
$$= h^2 + 8h + 15$$

1 point - evaluating $g(x)$ *for* $x = 4 + h$

1 point - simplifying

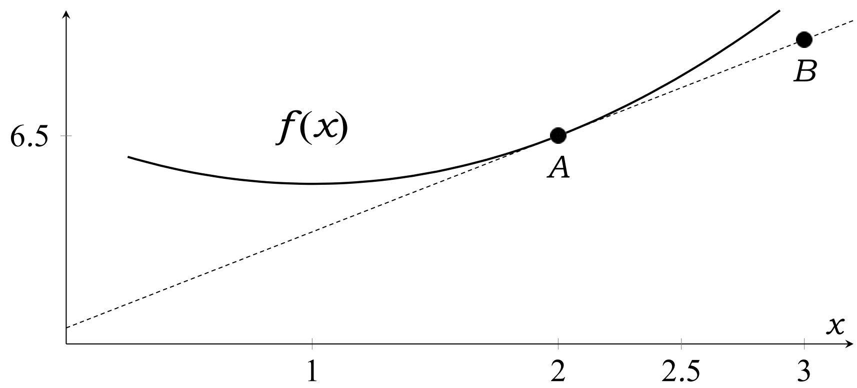
(c) [3 points] Use your work above to **find** $g^t(4)$ **using the limit definition** of the derivative.

$$g'(4) = \lim_{h\to 0} \frac{g(4+h) - g(4)}{h}$$
$$= \lim_{h\to 0} \frac{(h^2 + 8h + 15) - 15}{h} = \lim_{h\to 0} \frac{h^2 + 8h}{h} = \lim_{h\to 0}(h+8) = 8$$

2 points - writing the limit definition of g'(4)

1 point - using algebra to get to the answer

3. [6 points] The function in the figure has $f(2) = 6.5$ and $f'(2) = 3$.



(a) [3 points] Find the formula for the tangent line to $f(x)$ at $x = 2$.

$A - (2, 6.5)$

$m = f'(2) = 3$

$y - 6.5 = 3(x - 2) \rightarrow y = 3x + 0.5$

1 point – point A is on the tangent line
1 point - f'(2) is the slope of the tangent line
1 point – writing the equation of the tangent line

(b) [2 points] Use the picture and your equation from part (a) to find the coordinates for point B. Present your answer as an (x, y) pair.

$B:\ x = 3 \rightarrow y = 3 \times 3 + 0.5 = 9.5$

$B\ (3, 9.5)$

0.5 points for x coordinate
0.5 points for finding y-coordinate
1 point for giving the answer as (3, 9.5)

(c) [1 point] Use your work above to estimate $f(2.5)$. Write your answer as one number.

$f(2.5) \approx 3 \times 2.5 + 0.5 = 8$

4. [6 points] The wind speed $W(t)$ outside Madonna della Strada Chapel is measured once an hour over six consecutive hours.

Time (hours)	0	1	2	3	4	5	6
Wind (in knots)	31	21	16	13	5	7	21

(a) [3 points] Does $W^{t}(t)$ appear to be positive or negative during the interval $[0, 3]$? (Explain.)

$W'(t)$ is negative on [0, 3] - 1.5 points
because the values of $W(t)$ are decreasing on this interval - 1.5 points

(b) [3 points] Does $W^{tt}(t)$ appear to be positive or negative during the interval $[0, 3]$? (Explain.)

$W''(t)$ appears to be positive ($W(t)$ is concave up) - 1.5 points; because the amounts by which the values of $W(t)$ decrease in time are decreasing in value, **or** the values of $W'(t)$ are increasing – 1.5 points

5. [14 points] Find the requested derivatives. You are not required to simplify your final answer.

(a) [3 points] $h'(x)$ for $h(x)=\sqrt{x}\left(x+\sqrt{x}\right)$

1 point $\quad h(x)=x^{\frac{1}{2}}\left(x+x^{\frac{1}{2}}\right)=x^{\frac{3}{2}}+x$

2 points $\quad h'(x)=\frac{3}{2}x^{\frac{1}{2}}+1$

(b) [4 points] $p'(x)$ for $p(x)=2^{\sin(x^3)}$

$$p'(x)=(\ln 2)2^{\sin(x^3)}\cos\left(x^3\right)\left(3x^2\right)$$

1 point for each derivative of the three functions composed
1 point for p'(x)

(c) [4 points] $q'(x)$ for $q(x)=\dfrac{\ln(x)}{x+1}$

$$q'(x)=\frac{\frac{1}{x}(x+1)-\ln(x)\times(1)}{(x+1)^2}$$

2 points for using correctly the quotient rule
1 point for each derivative

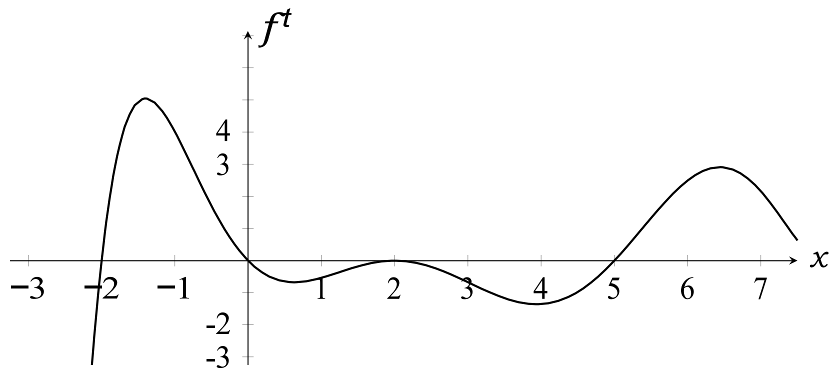
(d) [3 points] $f''(x)$ for $f(x)=x^7+9x+e^{3x}$

$$f'(x)=7x^6+9+3e^{3x}$$
$$f''(x)=42x^5+9e^{3x}$$

1 point for the derivative of power function, 1 point for the derivative of the exponential, 1 point for the right answer

6. [10 points] In this problem, we ask you to use information about f' to answer questions about a function f and its second derivative f''.

CAUTION: The graph below **depicts the derivative** of f.



(a) [2 points] Identify all **critical points** for f on the interval $(-3, 7)$.

- Zeros of f' $\to x = -2, 0, 2, 5$

(b) [2 points] Indicate which of the above, if any, correspond to **local maxima** for f.

f' changes sign form "+" to "-" ; $\to x = 0$

(c) [2 points] Which is larger, $f(6)$ or $f(7)$? (Explain.)

$f(7)$, $f' > 0$ *on* $[6, 7]$, thus f increases from $x = 6$ *to* $x = 7$

(d) [2 points] Which is larger, $f''(6)$ or $f''(7)$ (Explain.)

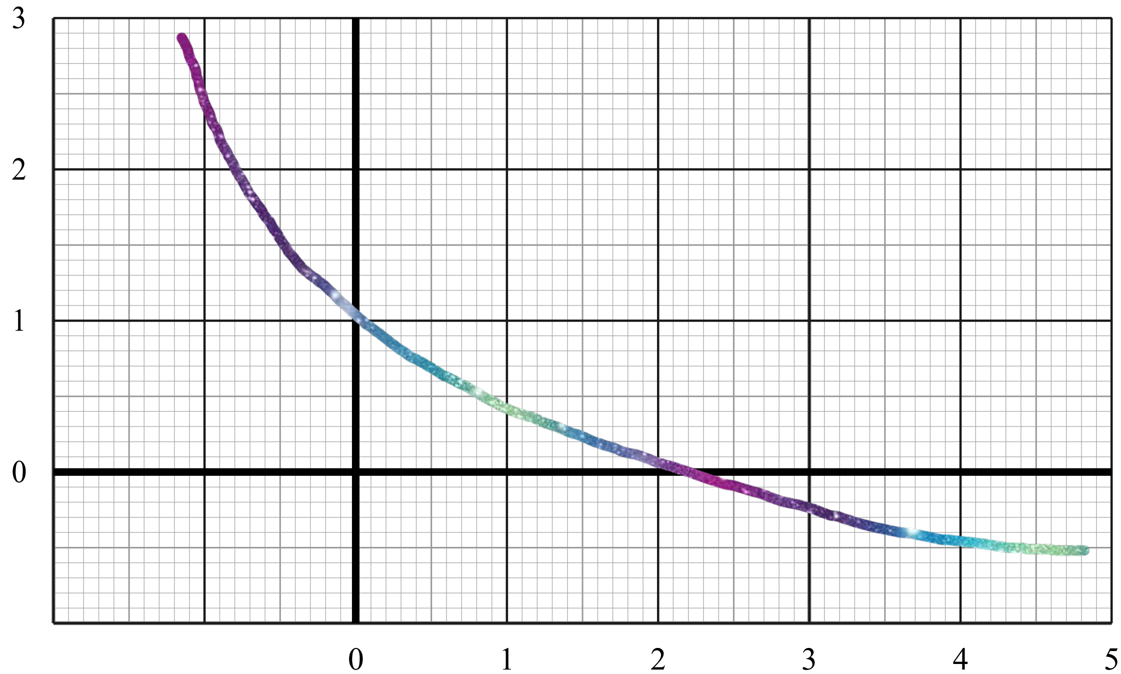
$f''(6)$, as $f''(6) > 0$ *and* $f''(7) < 0$

1 point for answer
1 point for explanation

(e) [2 points] Identify all **inflection points** for f on the interval $(-3, 7)$.

Points where $f'' = 0$ or f' has an extremum $\to x = -1.5, 0.5, 2, 4, 6.5$
(ok if -1.6, 0.6 or similar)

7. [4 points] Sketch the graph of a function $f(x)$ that is always decreasing, always concave up and satisfies $f(0) = 1$.



1 point for y-intercept y=1

1 point for always decreasing

2 points for always concave up

8. [10 points] A company that produces cell phones has a production capacity of up to 200 million units. The company estimates that the cost of producing a single cell phone varies with the production level and is determined by the cost function

$$C(x) = 0.05x^2 - 15x + 1625 \text{ for } 0 \le x \le 200,$$

where C is the cost (in dollars) of producing a single cell phone when the company has a level of production of x million cell phones.

(a) [2 points] Find a formula for $C'(x)$.

$$C'(x) = 0.1x - 15$$

(b) [3 points] Find the value of $C'(100)$ and write an interpretation for this value in the context of this problem. *Make sure to write your answer in a complete sentence and using the appropriate units.*

$C'(100) = 0.1 \times 100 - 15 = -5$ dollars per million units. – 1 point

The cost of producing one cell phone decreases by 5 dollars when the production increases from 100 million units to 101 million units. – 2 points

(c) [3 points] Find the critical points of $C(x)$ and classify them as local minima, local maxima, or neither.

$C'(x) = 0 \rightarrow x = 150 \textit{ million units}$ - 1 point

$C'(100) = -5,\ C'(200) = 5,\ C'$ changes sign from "-" to "+" at the critical point which is a minimum cost. – 2 points

(d) [2 points] What is the production level (in millions of cell phones) that gives the global minimum of the cost function C?

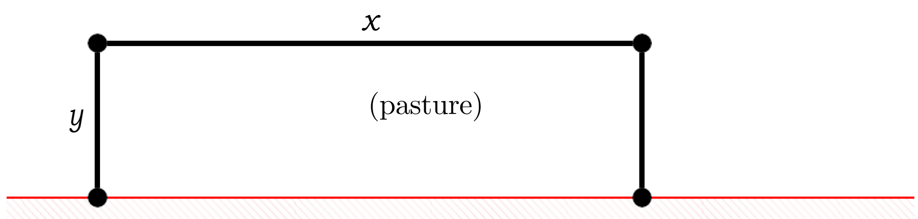
$C(0) = 1625 \textit{ dollars}$

$C(200) = 625 \textit{ dollars}$, **or** the cost function has only one critical point which is a local

$C(150) = 500 \textit{ dollars}$

minimum, thus it is also the global minimum at $150 \textit{ million units}$

9. [10 points] The livestock industry has determined that, to raise healthy cattle, a farm needs 20 square yards of space per cow. A small farmer is interested in acquiring 90 cows and needs to build a rectangular pasture that only requires three sides of fencing. (They will use one side of an already existing barn).



(a) [2 points] Write an equation involving x and y for the total length of new fencing (in feet) that needs to be installed to build a pasture for the cows.

$$l = x + 2y$$

(b) [3 points] Write an equation for the total length needed involving only x.

The area needed for 90 cows $A = 20\ yd^2 / cow \times 90\ cows = 1800\ yd^2$ - 1 point

Area of the rectangle $A = xy$

y as a function of x, $xy = 1800 \rightarrow y = \frac{1800}{x}$ - 1 point

Length of the fence as a function of x: $l = x + 2\frac{1800}{x}$; $l(x) = x + \frac{3600}{x},\ 0 < x < \infty$

- 1 point

(c) [5 points] What is the minimum length of fencing that needs to be purchased to build the pasture?

$$l(0) = \lim_{x\to\infty} l(x) = \infty \qquad -1\ pt$$

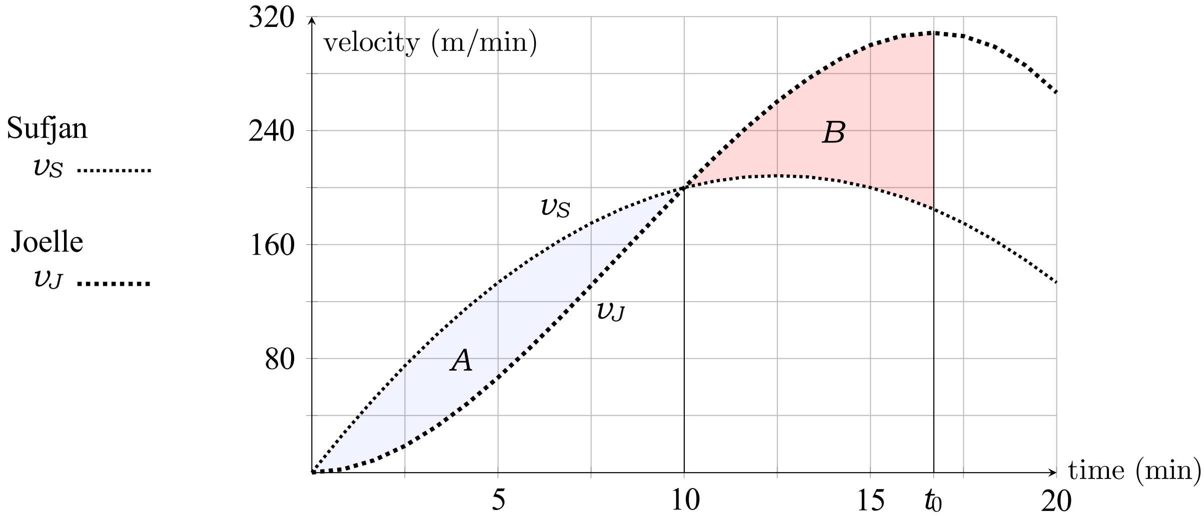
$$l'(x) = 1 - \frac{3600}{x^2}; \qquad -1\ pt$$

$$l'(x) = 0 \rightarrow 1 = \frac{3600}{x^2} \rightarrow x^2 = 3600 \rightarrow x = 60\ yd \qquad -2\ pt$$

$$l(60) = 60 + \frac{3600}{60} = 120\ yd\ = \min\ length \qquad -1\ pt.$$

10. [10 points] Sufjan and Joelle agreed to run a race for a local charity. Depicted below are the graphs of their velocities in meters per minute. (*e.g.*, 2.5 minutes after the race began, Sufjan was running at 80 meters per minute.)

At time t_0, the **shaded regions** A and B have **equal area**.
Suppose the winner finishes the race in 20 minutes.



(a) [2 points] True or False: Sufjan covered more ground than Joelle after 5 minutes.

True

(b) [2 points] True or False: Joelle caught up to Sufjan after 10 minutes.

False

(c) [2 points] True or False: Sufjan was A meters ahead of Joelle after 10 minutes.
True

(d) [2 points] True or False: Sufjan covered more ground than Joelle after 15 minutes.

True

(e) [2 points] Who won the race? (Justify your answer with a single sentence.)

Joelle. She caught up with Sufjan at $t = t_0$, and between that moment and $t = 20\ min$ she covered more ground that Sufjan.

11. [10 points] (a) [5 points] Compute the indefinite integral $\int\left(x^3+e^{3x}+\frac{1}{1+x^2}\right)dx$

$$\int\left(x^3+e^{3x}+\frac{1}{1+x^2}\right)dx=\frac{1}{4}x^4+\frac{1}{3}e^{3x}+\arctan(x)+C$$

1 point for using the addition property of integrals
1 point for each antiderivative
1 point for adding the constant C

(b) [5 points] Find the antiderivative $F(x)$ for $f(x)=x^3+6x$ that satisfies the property $F(2)=6$.

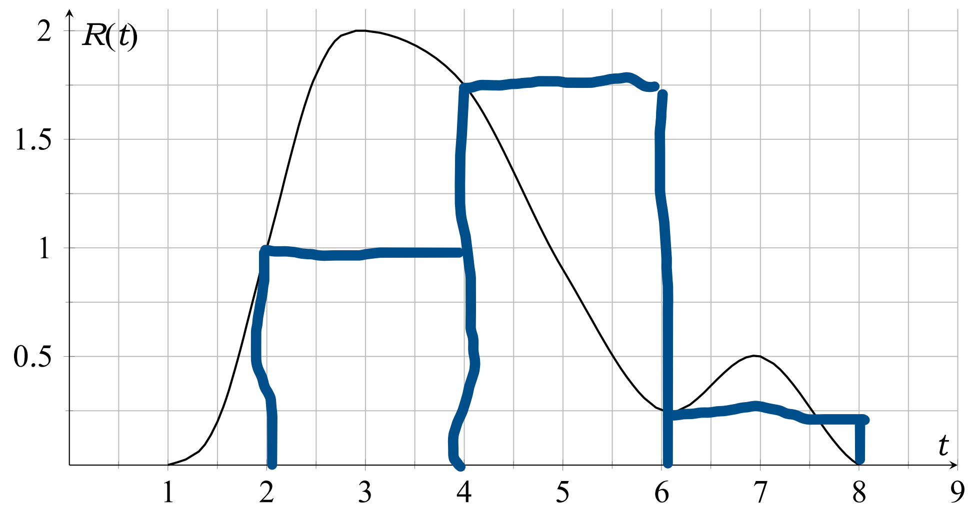
$$\int f(x)\,dx=\int\left(x^3+6x\right)dx=\frac{1}{4}x^4+6\times\frac{1}{2}x^2+C$$

$$F(2)=6\ \rightarrow 6=\frac{1}{4}\times 2^4+3\times 2^2+C\ \rightarrow C=-10$$

$$So\ \ F(x)=\frac{1}{4}x^4+6\times\frac{1}{2}x^2-10$$

2 points for the family of antiderivatives
2 points for finding C
1 points for the required antiderivative F(x)

12. [8 points] The figure below shows the rate R of snowfall (in inches per hour) during a recent winter storm in Chicago, t hours after midnight.



(a) [5 points] Estimate $\int_2^8 R(t)\,dt$ using a left Riemann sum with 3 subdivisions.

Interval [2, 8], $n = 3 \rightarrow \Delta t = \frac{8-2}{3} = 2\ hours$ - 1 point

$$LH - RS = R(2)\times 2 + R(4)\times 2 + R(6)\times 2$$
$$= 1\times 2 + 1.75\times 2 + 0.25\times 2 = 6\ inch$$

2 points for drawing the rectangles corresponding to LH-RS, or writing the sum
2 points for finding the value of LH- RS

(b) [3 points] Interpret $\int_2^8 R(t)\,dt$ in the context of this question. *Make sure to write your answer in a complete sentence with units. Use the value found in* (a) *as part of your answer.*

Between 2 am and 8 am the amount of snow fall was $\int_2^8 R(t)\,dt \approx 6\ inch$

Elementary Tools from Algebra and Geometry

Quadratic Formula: $ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Pythagorean Theorem: If a right triangle has legs a, b and hypotenuse c, then $a^2 + b^c = c^2$.

Triangle Area = $\frac{1}{2}$ base × height.

Rectangle Area = base × height

Circle Area = πr^2

Circle Perimeter = $2\pi r$

Perimeter of a polygon (triangle, rectangle, etc.) = sum of side lengths

Five derivative rules for operations on functions.

Constant Multiple Rule: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

Sum and Difference Rule: $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}\big(f(x) \cdot g(x)\big) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain Rule: $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$

Ten derivative rules for functions

Derivative of a Constant: $\frac{d}{dx}\big(c\big) = 0$, where c is a constant.

The Power Rule: $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$

Exponential Functions: $\frac{d}{dx}\big(a^x\big) = a^x \cdot \ln(a)$

Special Case: $\frac{d}{dx}\big(e^x\big) = e^x$

Three Trigonometric Rules:

$\frac{d}{dx}\big(\sin(x)\big) = \cos(x)$

$\frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$

$\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x) = \frac{1}{\cos^2(x)}$

Three Inverse Function Rules:

$\frac{d}{dx}\big(\ln(x)\big) = \frac{1}{x}$

$\frac{d}{dx}\big(\arctan(x)\big) = \frac{1}{1+x^2}$

$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}$

General Antiderivative Rules

If k is a constant $\int k\,dx = kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, when $n \neq -1$

$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

$\int e^x\,dx = e^x + C$

$\int \cos(x)\,dx = \sin(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$

$\int \frac{1}{x}\,dx = \ln(|x|) + C$

$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$